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Error estimates for a sixth-order theory of plate bending

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Abstract. A refined linear theory for the bending of anisotropic, homogeneous plates which takes account of transverse shear deformation and transverse normal stress is rigorously validated by imbedding it in the linear theory of elasticity. Three-dimensional displacement and stress fields are constructed from the two-dimensional plate theory and shown by the hypersphere theorem to approximate exact elasticity solutions with a relative mean square error proportional to the plate thickness cubed. This improves previous estimates for sixth-order theories involving error bounds proportional to the square of thickness.

1. Introduction

Engineering plate theories reduce the behavior of three-dimensional (3D) bodies to 2D equations. Practically, however, a knowledge of 3D displacement and stress distributions is essential. An effective tool for rationally reconstructing 3D information from given 2D plate theory quantities is offered by the hypersphere theorem of Prager and Synge [1, 2]. The theorem suggests that exact linear elasticity solutions can be approached by means of statically and kinematically admissible solutions and bounds their error in a mean square sense. Nordgren [3] applied this method to classical Kirchhoff's plate theory [4] with fourth-order differential field equations and obtained a relative error proportional to the plate thickness, $O(h)$. Simmonds [5] and Nordgren [6] refined this estimate to $O(h^2)$ by properly accounting for transverse shear deformation. The same-order error was found in [6] for Reissner's [7] sixth-order theory, in conflict with expectations that this higher-order theory should guarantee a better accuracy than Kirchhoff's lower-order theory. Berdichevskii [8] estimated the error of Reissner's theory at $O(h^3)$ but only in the case of plates with no surface loads. The present writer [9, 10] provided error bounds for Panc's [11] "component" fourth-order theory and Reissner's theory assuming materials with high transverse shear deformability.

This paper studies the accuracy of the recent sixth-order plate theory due to Rehfield and Valisetty [12]. The theory attracts attention because of its relative simplicity combined with unexpectedly high precision revealed in numerical examples, even for thick plates. Of special note is also the discussion in [12] of various nonclassical effects in 3D displacement and stress fields derived in [12] to accompany the 2D plate theory. Guided by those results, we verify Rehfield–Valisetty's theory by means of the hypersphere theorem, finding that its relative mean square error is proportional to the plate thickness cubed, thus generalizing or improving known estimates [6, 8, 10] for sixth-order theories. This also proves that such theories surpass in accuracy the classical fourth-order theory in the interior plate domain, not only near the edge.

In Sec. 2, we formulate the plate problem within the confines of 3D elasticity. Sec. 3 introduces the hypersphere theorem and some useful inequalities. In Sec. 4, the Rehfield–Valisetty plate theory [12] is recorded in a modified setting, better suited for our purposes. Sec. 5 presents appropriate 3D displacement and stress fields constructed from the 2D theory, some components of stress being more elaborate than in [12]. Sec. 6, finally, uses the hypersphere theorem to establish error estimates for the 3D fields of Sec. 5 with respect to exact elasticity solutions.

2. Three-dimensional plate problem

Our subject are plates of constant thickness $2h$ with upper and lower faces $z = \pm h$ sharing in equal parts a distributed lateral load p. Accordingly, the traction boundary conditions at the faces are

$$\sigma_{i3}(x_j, z = \pm h) = 0, \quad \sigma_{33}(x_j, z = \pm h) = \pm\tfrac{1}{2}p(x_j), \tag{1a, b}$$

where the $\sigma_{..}$ denote components of stress, $x_j (j = 1, 2)$ are cartesian coordinates in the middle plane and $x_3 = z$ is distance from that plane. Such loading conditions are antisymmetric with respect to the midsurface $z = 0$ and, consequently, only bending deformations result. General loads may always be split into symmetric and antisymmetric parts with corresponding stretching and bending problems solved separately.

On the cylindrical edge surface S with unit normal $\mathbf{n}$, the conditions are

$$\boldsymbol{\sigma} \cdot \mathbf{n} = \boldsymbol{\sigma}^* \cdot \mathbf{n} \text{ on } S_\sigma, \quad \mathbf{u} = \mathbf{u}^* \text{ on } S_u, \tag{2a, b}$$

S_σ and S_u being complementary parts of S with prescribed stresses $\boldsymbol{\sigma}^*$ and displacements $\mathbf{u}^*$.

The linear equations of equilibrium read

$$\sigma_{ij,j} + \sigma_{i3,3} = 0, \quad \sigma_{i3,i} + \sigma_{33,3} = 0, \tag{3a, b}$$

where body forces are zero, a comma denotes partial differentiation with respect to x_i and x_3 and summation over repeated indices is assumed over the range 1, 2.

The plate is taken to be homogeneous, linearly elastic and anisotropic, with elastic symmetry relative to the midplane. The constitutive equations thus have the form

$$\sigma_{ij} = D_{ijkl}u_{(k,l)} + C_{ij}\sigma_{33}, \tag{4}$$

$$\sigma_{i3} = B_{i3j3}(u_{j,3} + u_{3,j}), \tag{5}$$

$$\sigma_{33} = B_{33ij}u_{(i,j)} + B_{3333}u_{3,3}, \tag{6}$$

where

$$D_{ijkl} = B_{ijkl} - B_{ij33}B_{33kl}/B_{3333}, \tag{7}$$

$$C_{ij} = B_{ij33}/B_{3333}. \tag{8}$$

Here the $B_{....}$ are components of the elasticity tensor and they give rise to two auxiliary tensors D_{ijkl} and C_{ij}; the $u_{.}$ are components of displacement and a pair of subscripts enclosed in parentheses indicates symmetrization.

3. Hypersphere theorem

The elastic energy functional

$$\|\boldsymbol{\sigma}\|^2 = \int_{-h}^{h} \int_F (A_{ijkl}\sigma_{ij}\sigma_{kl} + 4A_{i3j3}\sigma_{i3}\sigma_{j3} + 2A_{ij33}\sigma_{ij}\sigma_{33} + A_{3333}\sigma_{33}\sigma_{33})\, \mathrm{d}F\, \mathrm{d}z, \tag{9}$$

where the $A_{....}$ are the reciprocals of the $B_{....}$ in (4)–(8), F is the region of the midplane and $\boldsymbol{\sigma}$ denotes, for the present, an arbitrary stress field, is quadratic, homogeneous and positive definite; consequently, it defines a norm for stress.

The hypersphere theorem [1, 2] asserts that

$$\|\boldsymbol{\sigma} - \tfrac{1}{2}(\boldsymbol{\sigma}' + \boldsymbol{\sigma}'')\| / \|\boldsymbol{\sigma}''\| = \tfrac{1}{2}e \tag{10}$$

where

$$e = \|\boldsymbol{\sigma}' - \boldsymbol{\sigma}''\| / \|\boldsymbol{\sigma}''\|. \tag{11}$$

It follows from (10) and (11) that an exact solution $\boldsymbol{\sigma}$ to the 3D plate problem (1)–(6) may be approximated by $(\boldsymbol{\sigma}' + \boldsymbol{\sigma}'')/2$, the corresponding relative mean square error being computed from (11), where $\boldsymbol{\sigma}'$ and $\boldsymbol{\sigma}''$ are statically and kinematically admissible stress fields, respectively.

The following inequalities:

$$\|\boldsymbol{\sigma}' - \boldsymbol{\sigma}\| / \|\boldsymbol{\sigma}''\| \leqq e, \quad \|\boldsymbol{\sigma}''(\mathbf{u}'') - \boldsymbol{\sigma}(\mathbf{u})\| / \|\boldsymbol{\sigma}''(\mathbf{u}'')\| \leqq e, \tag{12a, b}$$

provide error bounds when $\boldsymbol{\sigma}'$ and $\boldsymbol{\sigma}''$ are used separately as approximations to $\boldsymbol{\sigma}$. Additionally, (12b) may be interpreted as an error estimate for kinematically admissible displacements $\mathbf{u}''$ with respect to exact displacements $\mathbf{u}$, provided that rigid-body motions are precluded by support conditions.

Our task in the sections which follow is to construct appropriate 3D distributions of $\boldsymbol{\sigma}'$, $\mathbf{u}''$ and $\boldsymbol{\sigma}''$ given the 2D equations of Rehfield–Valisetty's plate theory [12]. Ideally, $\boldsymbol{\sigma}'$ should satisfy the equilibrium equations (3) and traction boundary conditions (1) and (2a), whereas $\mathbf{u}''$ and $\boldsymbol{\sigma}''$ ought to fulfil the constitutive equations (4)–(6) and the displacement boundary conditions (2b). To render the problem tractable, however, $\boldsymbol{\sigma}'$ and $\mathbf{u}''$ are sought, see [3, 5, 6, 8–10, 13], with no regard to the boundary conditions (2) on the edge surface S, thereby admitting only such edge tractions $\boldsymbol{\sigma}^*$ and displacements $\mathbf{u}^*$ that conform to $\boldsymbol{\sigma}'$ and $\mathbf{u}''$, i.e.,

$$\boldsymbol{\sigma}^* \cdot \mathbf{n} = \boldsymbol{\sigma}' \cdot \mathbf{n} \text{ on } S_\sigma, \quad \mathbf{u}^* = \mathbf{u}'' \text{ on } S_u. \tag{13}$$

Deviations from these so-called "regular boundary conditions", cf. [13], give rise to an error in addition to that in (11). Practically, such deviations are expected to be small and may be accounted for as in [13].

4. Two-dimensional plate theory

The theory under consideration involves 2D kinematic variables defined from the 3D kinematically admissible displacements $\mathbf{u}''$ as

$$w(x_j) = u_3''(x_j, z = 0), \quad b_i(x_j) = u_{i,3}''(x_j, z = 0), \tag{14}$$

w and b_i thus being the lateral deflection and rotations of the midsurface.

The static variables encompass moments M_{ij} and shear forces Q_i which are related to the 3D statically admissible stresses $\boldsymbol{\sigma}'$ by

$$M_{ij}(x_k) = \int_{-h}^{h} \sigma_{ij}'(x_k, z) z \, \mathrm{d}z, \quad Q_i(x_k) = \int_{-h}^{h} \sigma_{i3}'(x_k, z) \, \mathrm{d}z. \tag{15a, b}$$

The overall equilibrium equations read

$$M_{ij,j} = Q_i, \quad Q_{i,i} = -p. \tag{16a, b}$$

The moment constitutive equations are

$$M_{ij} = D_{ijkl}(\tfrac{2}{3}h^3 b_{(k,l)} + \tfrac{2}{5}h^2 f_{(k,l)}) + \tfrac{2}{5}h^2 C_{ij} p, \tag{17}$$

where

$$b_i = -w_{,i} + \frac{3}{h} A_{i3j3} T_j, \tag{18}$$

$$f_i = -\tfrac{1}{6}h^3 C_{jk} w_{,jki} - A_{i3j3} T_j, \tag{19}$$

$$T_i = -\tfrac{2}{3}h^3 D_{ijkl} w_{,klj}, \tag{20}$$

and, by definition of A_{i3j3},

$$4A_{i3j3} B_{i3k3} = \delta_{jk}, \tag{21}$$

δ_{ij} being the Kronecker delta.

Use of the (18)–(20) gives the moments in (17) in terms of the deflection w,

$$M_{ij} = -D_{ijkl}(\tfrac{2}{3}h^3 w_{,kl} + \tfrac{1}{15}h^5 C_{ac} w_{,ackl}) - \tfrac{16}{15}h^5 D_{ijkl} D_{acrs} A_{(k3a3} w_{,rscl)} + \tfrac{2}{5}h^2 C_{ij} p. \tag{22}$$

Introducing (22) into (16a) yields the shear forces

$$Q_i = -D_{ijkl}(\tfrac{2}{3}h^3 w_{,klj} + \tfrac{1}{15}h^5 C_{ac} w_{,acklj}) - \tfrac{16}{15}h^5 D_{ijkl} D_{acrs} A_{(k3a3} w_{,rscl)j} + \tfrac{2}{5}h^2 C_{ij} p_{,j}. \tag{23}$$

Substitution of (23) into (16b) results in a sixth-order differential equation for w of the form

$$D_{ijkl}(\tfrac{2}{3}h^3 w_{,klji} + \tfrac{1}{15}h^5 C_{ac} w_{,ackljі}) + \tfrac{16}{15}h^5 D_{ijkl} D_{acrs} A_{(k3a3} w_{,rscl)ji} = p + \tfrac{2}{5}h^2 C_{ij} p_{,ji}. \quad (24)$$

Appropriate boundary conditions to be used in conjunction with this equation are

$$M_{ij}n_j = M_{ij}^* n_j, \quad Q_i n_i = Q_i^* n_i \text{ on } C_\sigma, \quad (25)$$

and

$$w = w^*, \quad b_i = b_i^* \quad \text{on } C_u, \quad (26)$$

where C_σ and C_u denote complementary parts of the edge curve C of the midplane with prescribed static and geometric quantities, respectively. By virtue of (18), (20), (22) and (23) the above conditions may be expressed through the basic unknown w.

Disregarding unimportant differences in notation, the above plate-theory equations are effectively those of Rehfield and Valisetty [12]. For later convenience, we have expressed all variables in terms of the lateral deflection w. Additionally, in contrast to [12], stretching deformation has not been taken into account for the sake of simplicity, but this entails no loss of generality since the stretching and bending problems are not coupled and their solutions may be simply superimposed.

5. Three-dimensional displacements and stresses

Consider the following displacement field, with the notation $t = z/h$,

$$u_i''(x_j, t) = thb_i(x_j) + t^3 f_i(x_j), \quad (27)$$

$$u_3''(x_j, t) = w(x_j) + t^2 g(x_j) + t^4 r(x_j) + (6t^2 - t^4) q(x_j), \quad (28)$$

where

$$g = -\tfrac{1}{2}h^2 C_{ij} b_{(i,j)}, \quad r = -\tfrac{1}{4}h C_{ij} f_{(i,j)}, \quad (29a, b)$$

$$q = hp/16B_{3333}, \quad (30)$$

and two stress fields

$$\sigma_{ij}'' = D_{ijkl}(thb_{(k,l)} + t^3 f_{(k,l)}) + \tfrac{1}{4}(3t - t^3) C_{ij} p, \quad (31)$$

$$\sigma_{i3}'' = \tfrac{3}{4}h(1 - t^2) T_i - \tfrac{3}{2}t^2 h B_{i3j3} C_{rs} A_{(r3a3} T_{a,s)j} + t^4 B_{i3j3} r_{,j} + (6t^2 - t^4) B_{i3j3} q_{,j}, \quad (32)$$

$$\sigma_{33}'' = \tfrac{1}{4}(3t - t^3) p, \quad (33)$$

and

$$\sigma'_{ij} = (3t/2h^2)M_{ij} + \tfrac{1}{20}(3t - 5t^3)(C_{ij}p - 4D_{ijkl}f_{(k,l)}), \tag{34}$$

$$\sigma'_{i3} = \tfrac{3}{4}h(1 - t^2)Q_i + \frac{h}{20}(1 - 6t^2 + 5t^4)(\tfrac{1}{4}C_{ij}p_{,j} - D_{ijkl}f_{(k,l)j}), \tag{35}$$

$$\sigma'_{33} = \tfrac{1}{4}(3t - t^3)p - (h^2/20)(t - 2t^3 + t^5)(\tfrac{1}{4}C_{ij}p_{,ji} - D_{ijkl}f_{(k,l)ji}). \tag{36}$$

It is readily seen that these 3D distributions are specified in terms of the 2D plate theory presented in Sec. 4 and are consistent with relations (14) and (15) defining the static and kinematic variables of that theory. In view of (18), (19), (21), (29) and (30), the displacements $\mathbf{u}''$ in (27), (28) and stresses $\boldsymbol{\sigma}''$ in (31)–(33) satisfy the constitutive equations (4)–(6), thus being kinematically admissible. By virtue of (16), the stresses $\boldsymbol{\sigma}'$ in (34)–(36) fulfil the traction boundary conditions (1) on the faces and equilibrium equations (3) and, consequently, constitute a statically admissible stress field. The next section also shows that $\boldsymbol{\sigma}'$ approaches $\boldsymbol{\sigma}''$ very closely.

The above fields are in essence equivalent to those originally found by Rehfield and Valisetty [12]. For our purposes, however, it has been necessary to make a clear distinction between statically and kinematically admissible quantities. In particular, the stresses in (35) and (36) are more elaborate than in [12] to render $\boldsymbol{\sigma}'$ in (34)–(36) statically admissible.

It is worth mentioning that $\mathbf{u}''$, $\boldsymbol{\sigma}''$ and $\boldsymbol{\sigma}'$ as given above incorporate the nonclassical effects of transverse shear and normal strains, transverse normal stresses and self-equilibrating over the thickness, all of these contributions being essential for having the refined error estimates we are now proceeding to establish.

6. Error estimates for three-dimensional solutions

From (31)–(36) and (17), the difference $\boldsymbol{\sigma}' - \boldsymbol{\sigma}''$ is found to have components

$$\sigma'_{ij} - \sigma''_{ij} = 0, \tag{37}$$

$$\begin{aligned}\sigma'_{i3} - \sigma''_{i3} = {} & \tfrac{3}{4}h(1 - t^2)(Q_i - T_i) - t^4 B_{i3j3}r_{,j} - (6t^2 - t^4)B_{i3j3}q_{,j} \\ & + \tfrac{3}{2}t^2 hB_{i3j3}C_{rs}A_{(r3a3}T_{a,s)j} + \frac{h}{20}(1 - 6t^2 + 5t^4)(\tfrac{1}{4}C_{ij}p_{,j} - D_{ijkl}f_{(k,l)j}),\end{aligned} \tag{38}$$

$$\sigma'_{33} - \sigma''_{33} = -(h^2/20)(t - 2t^3 + t^5)(\tfrac{1}{4}C_{ij}p_{,ji} - D_{ijkl}f_{(k,l)ji}). \tag{39}$$

This error stress field is expressible through the lateral deflection w, using (20) and relations

$$f_i = -\tfrac{1}{6}h^3 C_{jk}w_{,jki} + \tfrac{2}{3}h^3 A_{i3a3}D_{ajkl}w_{,klj}, \tag{40}$$

$$Q_i - T_i = -\tfrac{1}{15}h^5 D_{ijkl}(C_{ac}w_{,acklj} + 16D_{acrs}A_{(k3a3}w_{,rscl)j}) + \tfrac{4}{15}h^5 C_{ij}D_{arsc}w_{,scraj} + O(h^7), \tag{41}$$

$$p = \tfrac{2}{3}h^3 D_{ijkl} w_{,klji} + O(h^5), \tag{42}$$

$$q = (D_{ijkl}/24B_{3333})h^4 w_{,ijkl} + O(h^6), \tag{43}$$

$$r = \tfrac{1}{24}h^4 C_{ic} C_{jk} w_{,jkic} - \tfrac{1}{6}h^4 C_{ic} A_{(i3a3} D_{ajkl} w_{,kljc)}, \tag{44}$$

which follow from (19), (20), (24), (29b) and (30). Then by (20) and (40)–(44), the components of $\boldsymbol{\sigma}' - \boldsymbol{\sigma}''$ in (37)–(39) may be evaluated as

$$\sigma'_{ij} - \sigma''_{ij} = 0, \quad \sigma'_{i3} - \sigma''_{i3} = O(h^4), \quad \sigma'_{33} - \sigma''_{33} = O(h^5), \tag{45}$$

where, for simplicity, only h-dependence has been exposed. Likewise, from (31)–(33) with (18)–(20) and (42)–(44) it follows that

$$\sigma''_{ij} = O(h), \quad \sigma''_{i3} = O(h^2), \quad \sigma''_{33} = O(h^3). \tag{46}$$

Now from (9), after integrating over the thickness, the norms of the stresses in (45) and (46) have estimates

$$\|\boldsymbol{\sigma}' - \boldsymbol{\sigma}''\|^2 = O(h^9), \quad \|\boldsymbol{\sigma}''\|^2 = O(h^3). \tag{47}$$

On this basis we finally conclude that the relative mean square error e to be found from (11) will with (47) assume the form

$$e = h^3/L^3 + O(h^n), \quad n > 3. \tag{48}$$

Here L has been introduced to ensure a dimensionless character of e. Physically, L may be interpreted as a mean square wave length characterizing the midsurface deformation pattern in terms of its lateral deflection w. Since L and e become exceedingly complex when expressed through w, we will not record them.

Practically, given a particular plate problem we solve for w the corresponding 2D plate-theory equations in Sec. 4. Knowing w, one obtains from (27)–(36) 3D displacement and stress distributions $\mathbf{u}''$, $\boldsymbol{\sigma}''$ and $\boldsymbol{\sigma}'$. These fields are then introduced into (11) to give, after integration throughout the plate volume, the error e which by virtue of the hypersphere theorem (10) and inequalities (12) characterizes the closeness of $\mathbf{u}''$, $\boldsymbol{\sigma}''$ and $\boldsymbol{\sigma}'$ to the exact (unknown) 3D elasticity solution $\mathbf{u}$ and $\boldsymbol{\sigma}$. Besides this, one should verify whether the regular boundary conditions (13) on the edge surface are met. If not, irregular displacements $\mathbf{u}^* - \mathbf{u}''$ and stresses $\boldsymbol{\sigma}^* - \boldsymbol{\sigma}'$ produce errors in addition to e in (11). For more detail about such edge-zone contributions we refer to [13], observing only here that $\boldsymbol{\sigma}^* - \boldsymbol{\sigma}'$ represents a self-equilibrating stress distribution that, by virtue of Saint-Venant's principle, is expected to attenuate from the edge.

The physical significance of our result may be assessed by noting that for an isotropic plate under uniform face load, L is roughly the order of the plate width. Consequently, even for a fairly thick plate, say $h/L = 1/3$, the error is from (48) $e = (1/3)^3 \approx 3\%$, meaning that predictions of Rehfield–Valisetty's theory are very accurate. This has been demonstrated in [12] in an example and the present analysis confirms that observation with rigor and generality.

The fact that the error e is proportional to the plate thickness cubed, $O(h^3)$, is the main novel finding of this report and it represents a significant improvement over previous estimates [3, 5, 6, 8–10]. As a by-product, it has now been firmly established that transverse normal stress is important in the moment constitutive equations (17) of plate theory. Also, it follows from our analysis that sixth-order theories offer a better accuracy than fourth-order ones in the interior plate domain, in addition to their well-known superiority near the edges. This conclusion is verified by noting that in order to have a fourth-order differential equation for w, the moments (17) must be simplified by dropping f_i and taking $b_i = -w_{,i}$ instead of (18). Then only such $\mathbf{u}''$, $\boldsymbol{\sigma}''$ and $\boldsymbol{\sigma}'$ can be found at best where e is proportional to the square of thickness.

7. Conclusions

The hypersphere theorem has been applied to rigorously prove that Rehfield–Valisetty's simple, sixth-order theory predicts very accurately the 3D behavior of plates in bending. Adequate displacement and stress distributions throughout the plate have been provided and estimated to have a relative mean square error proportional to the cube of plate thickness with respect to exact elasticity solutions. This strongly supports expectations based on numerical studies [12] that sixth-order theories are appropriate for modelling of thick plates with rapidly fluctuating surface loads.

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